Innovations in Risk Analytics

Calibrating Trees to the Market Price of Options

Authors:
Jim Clark
Les Clewlow
Chris Strickland

Lacima Group

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In this article, we outline an approach for calibrating a local volatility surface single factor model to market prices of average price (Asian) options. Because of the numerical algorithm used to implement these models, they are often called 'implied trees'. A further article will show how to value exotic options, using the calibrated local volatility surface, consistent with the market prices.

The basic idea of implied tree approaches is to construct a trinomial tree[1] that is consistent with, or 'fits' currently traded energy derivatives prices, whether exactly or to some approximation. Once the local volatility function is determined, the future evolution of the energy price is known. The trinomial tree can then be used to price any other derivative on the same underlying energy asset with the same or earlier maturity secure in the knowledge that our pricing model is completely consistent with the prices of all liquid options with the same underlying commodity. Trinomial trees are a convenient way of valuing option contracts with complex, path dependent payoffs such as for indexed Swing options [Breslin et. Al.]. Market implied information embodied in the constructed tree thus enables the pricing of OTC and exotic options on the same underlying process.

Previous work by, for example, Derman, Kani, and Chriss [1996], describes the building of trees in terms of the spot price, consistent with a Geometric Brownian Motion process, with using market information that is embedded in standard European call and put options. In many energy markets, it is common place to implement models more generally applicable to the special characteristics of commodity prices (such as models with mean reverting drift terms), and where the liquid option contracts are typically Asian or average price options on monthly futures contracts, and in this article we make these extensions.

Option prices, such as for Asians are quoted in terms of an implied volatility. This is the volatility which makes a simple Black based option formula equal to the market price. The implied volatility varies with the time to maturity of the option which reflects the time varying volatility of the underlying commodity. It also varies with the strike price of the option which reflects the fact that the true distribution of the underlying commodity is fat-tailed relative to the lognormal distribution assumed by the Black model.

This variation in volatility as a function of the contract maturity and strike price (volatility surface) is commonly referred to as the volatility smile or skew. This "implied" methodology has the serious shortcomings that the model assumes a constant volatility and therefore the implied volatility surface is actually inconsistent with the underlying model assumed to price the options. This leads to the problem that there is no clear way to use these implied volatility surfaces to price more

[1] Binomial trees can be somewhat less convenient (or more restrictive) when calibrating to market data.

complex or exotic options consistent with the market quoted options. The fact is that the implied volatilities are not really volatilities at all but a "volatility-like" measure of the option price.

The popularity of Asian options over the last two decades has resulted in a large body of research focused on the valuation of these instruments. Some of the more popular techniques include those by Turnbull and Wakeman [1991], and Levy [1992]. Typically the models underpinning these valuation methods are approximations of the actual Asian contract specification and, as observed in the previous paragraph, mostly assume a constant volatility surface.

The process of calibrating a local volatility model to market option data comprises the following steps:

- **Market data:** Obtain market quotes for interest rate curve, futures contracts and options.

- **Construction of trinomial tree:** Constructing an approximate trinomial tree for the spot price based on a mean reverted time dependent volatility model for the spot price.

- **Calibrating the trinomial tree:** Market forward prices and Asian option prices obtained for a grid of various strikes and maturities are used to calibrate the trinomial tree and hence determine a local volatility surface.

Once the trinomial tree has been calibrated to the market data, it will contain all the relevant information (state prices and transition probabilities that reflect the local volatility surface) needed to price complex or exotic options. The pricing of such options will be the topic of discussion in a further article.

Construction of the Spot Price Tree

In order to construct a reasonable trinomial tree for the spot price we need an initial estimate of the spot price volatility and mean reverting dynamics. A reasonably realistic model for oil spot prices is the mean reverting spot price model first described by Schwartz [1997] and later extended by Clewlow and Strickland [1999, 2000] to be consistent with the market forward curve and spot volatilities. This model can be described by the following equation:

$$\frac{dS(t)}{S(t)} = \left[\frac{\partial \ln F(0,t)}{\partial t} + \alpha\left[\ln F(0,t) - \ln S(t)\right] + \frac{1}{2}\alpha\int_0^t \sigma(u)^2 \exp\left(-2\alpha(t-u)\right)du\right]dt + \sigma(t)dz(t) \tag{1}$$

Where $S(t)$ is the spot price at time t, $F(t,T)$ is the forward price at time t with maturity date T, α is the mean reversion rate, $\sigma(t)$ is the spot price volatility at time t, and $dz(t)$ is an increment in a Wiener process.

Under this model European call option prices are given by:

$$C(t, F(t,s); K, T, s)_{EuroCall} = P(t,T)\left(F(t,s)N(d_1) - KN(d_2)\right)$$

(2)

$$d_1 = \frac{\ln\left(F(t,s)/K\right) + \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}}$$

$$d_2 = d_1 - \tilde{\sigma}$$

$$\tilde{\sigma}^2 = \int_t^T \left(\sigma(u)\exp(-\alpha(s-u))\right)^2 du$$

where $\tilde{\sigma}^2$ is the integral of the forward return variance over the life of the option.

Using equation (2) and market quotes for standard European calls and/or puts the mean reversion rate and spot price volatility curve can be obtained by best fitting the model prices to the market prices[2].

Figure 1 shows a spot price volatility curve with seasonal variation obtained from a typical calibration of ATM call and put options. For the purposes of this article, we assume that interest rates are deterministic, and hence the modeling of a forward or futures contract will be treated as the same.

[2] If the market prices are for American options then an extension of the Barone-Adesi Whaley [1987] model which takes account of the mean reversion and time dependent spot price volatility can be used.

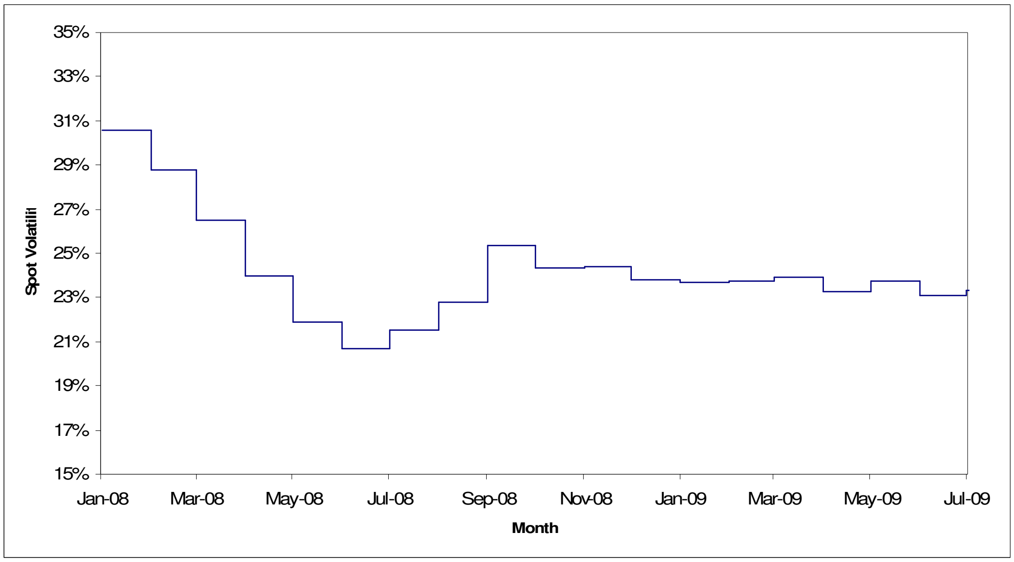


Figure 1: Spot price volatility curve

Once the spot mean reversion rate and volatility values have been determined, a trinomial tree is constructed to describe the distribution of the spot prices for different maturities (see Clewlow and Strickland (2000) for details). Figure 2 illustrates the trinomial tree structure.

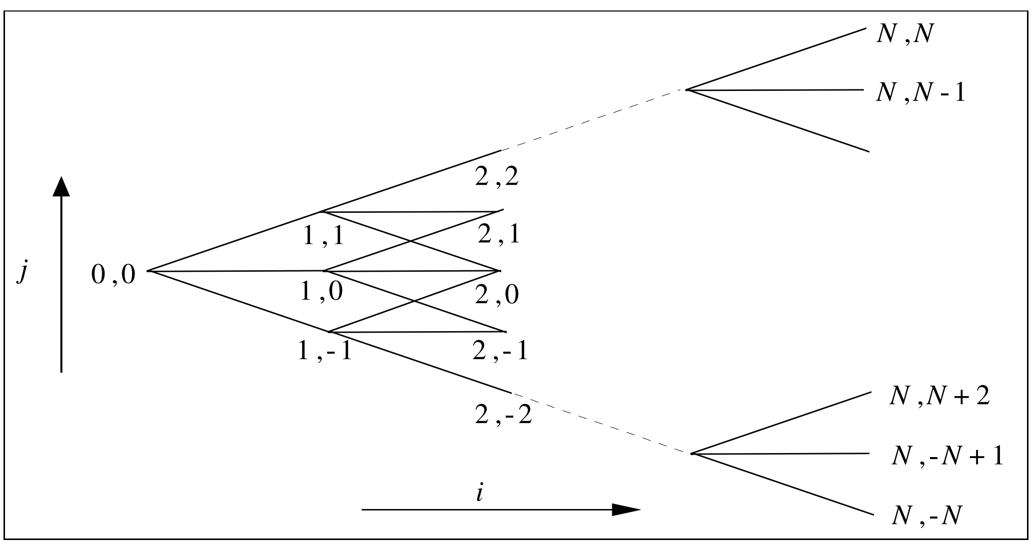


Figure 2: Spot price trinomial tree structure

Each node in the tree is identified by a pair of integers (i, j) where $i = 0,...,N$ is the time step and $j = -i,...,i$ is the level of the asset price relative to the initial asset price. At node (i, j) the date is t_i where we assume $t_0 = 0$ and the spot price is

$S_{i,j}$ with $S_{0,0} = S(0)$, the current spot price. Each node has three branches which connect with nodes at the next time step. We label these branches the up (u), middle (m) and down (d) branches. The up branch is the branch which goes to the highest spot price of the three branches at the next time step but does not necessarily go to a higher spot price level than the node from which it emanates. Similarly, the down branch goes to the lowest spot price of the three branches at the next time step and the middle branch is between the up and down branches. Each branch has an associated transition probability which defines the discrete time and state dynamics of the spot price. The tree is built with daily time steps in order to capture the calibration products accurately and in terms of the natural logarithm of the spot price.

Each node in the tree corresponds to a spot price. Further, corresponding to each spot price is a future forward curve required for obtaining the payoff of all options calibrated to, and priced off the trinomial tree. In order to capture the local volatility of market option quotes, and to be able to price an option off the tree, state (Arrow-Debreu) prices and transition probabilities must first be obtained.

Calibrating the trinomial tree

In order to capture the local volatility skew and smile of the market data, the trinomial tree needs to be calibrated to an appropriate set of market option quotes. For the local volatility model described in this article, calibration of the trinomial tree is with European style Asian calls and puts.

Calibrating the tree to market quotes is achieved through the use of state prices. These state prices provide a means to relate the various option market quotes at various strikes and maturities to transition probabilities (and hence local volatilities) at each corresponding node in the tree. Once both the state prices and transition probabilities have been obtained, vanilla and/or exotic options can be priced off the tree.

We calculate the state prices for each node such that the tree is consistent with the market forward prices and option prices. Define the state price $Q_{i,j}$ as the value, at time 0, of a security that pays 1 unit of cash if node (i, j) is reached, and zero otherwise. State prices are the building blocks of all securities; in particular, the price today ($t = 0$) $C(0)$ of any European claim with a payoff $C(S)$ as a function of the spot price S at time step i in the tree is given by:

$$C(0) = \sum_j Q_{i,j} C(S_{i,j})$$

where the summation takes place across all of the nodes j at time step i.

At each time step we have the following relationship between the forward price with maturity date at that time step and the state prices:

$$F(0,T_i) = \frac{1}{P(0,T_i)} \sum_j Q_{i,j} S_{i,j}$$

(4)

Where $P(0,T_i)$ is the price of a pure discount bond at time 0 which pays 1 unit of cash at time T_i and represents the discount factor from 0 to T_i. Figure 3 shows an example initial forward curve for WTI crude oil around the middle of November 2008, representing prices $F(0,T_i)$ that the trinomial tree is to be calibrated to.

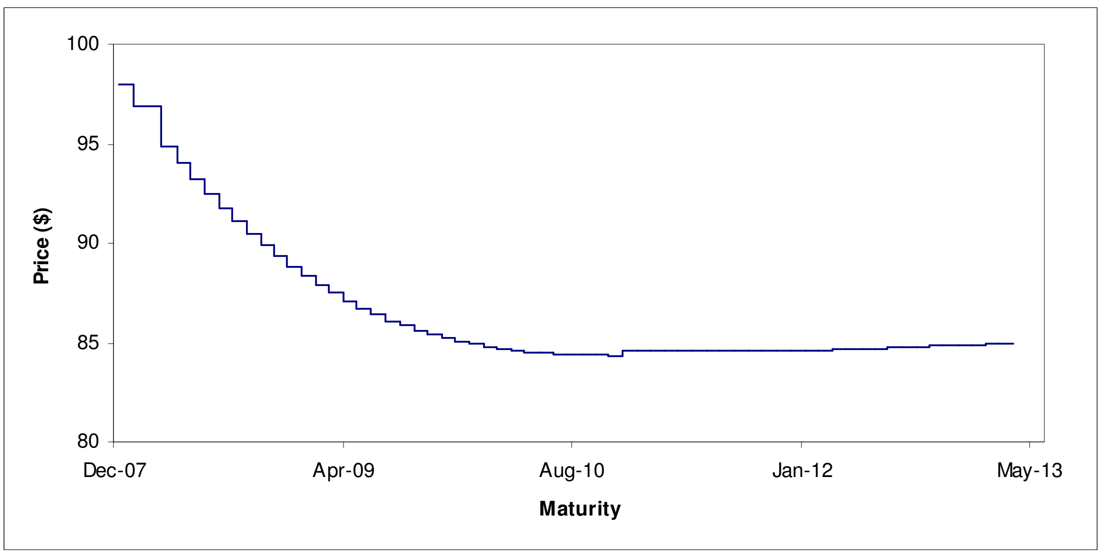


Figure 3: Initial forward curve for crude oil

The Asian options to which we calibrate are fixed on daily one month averages. The price of an Asian call option can be written as:

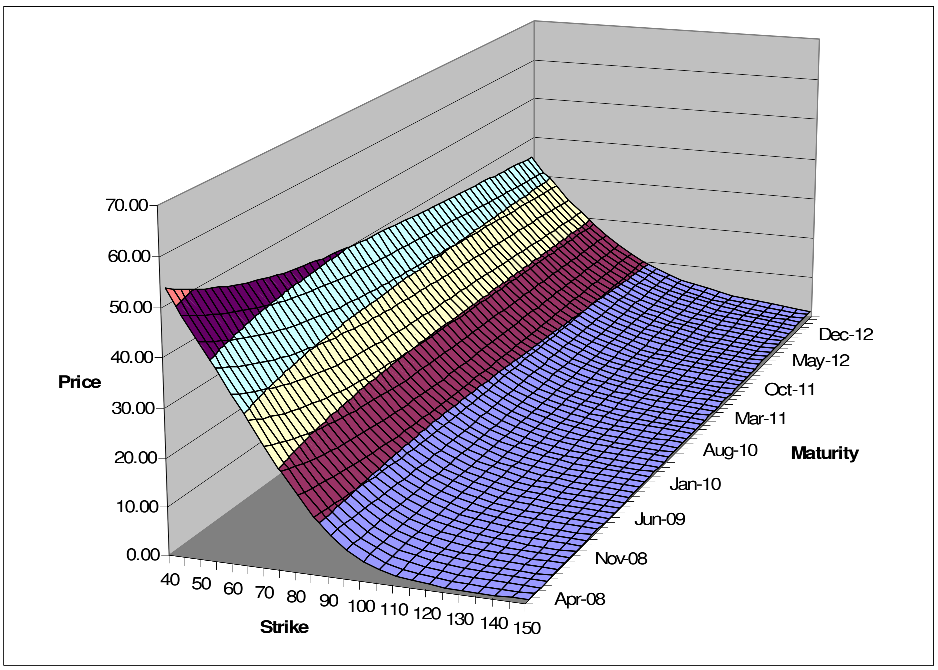
$$C(0,S_0;K,T)_{AsianCall} = \sum_j Q_{i,j} C(T_i,S_j;K,T)$$

(5)

Where $\sum_j$ is the summation over all nodes j at time step i, $Q_{i,j}$ is the state price for the node (i,j), $C(T_i,S_j;K,T)$ is the price of the Asian option at node (i,j) where T_i is the first fixing date of the Asian option. We can obtain an approximate value for $C(T_i,S_j;K,T)$ by using the approximate transition probabilities consistent with the mean reversion rate and spot volatility obtained earlier. For a particular Asian option first fixing date T_i, the state prices at each spot price level j can be determined by solving a system of linear equations of the form of equation (5) each for a different strike price. The required prices can be obtained by interpolating from the implied volatility surface for the observable market strikes.

Figures 4a and 4b show example Asian call and put option price surfaces that we calibrate to using the above methodology.

(a)



(b)

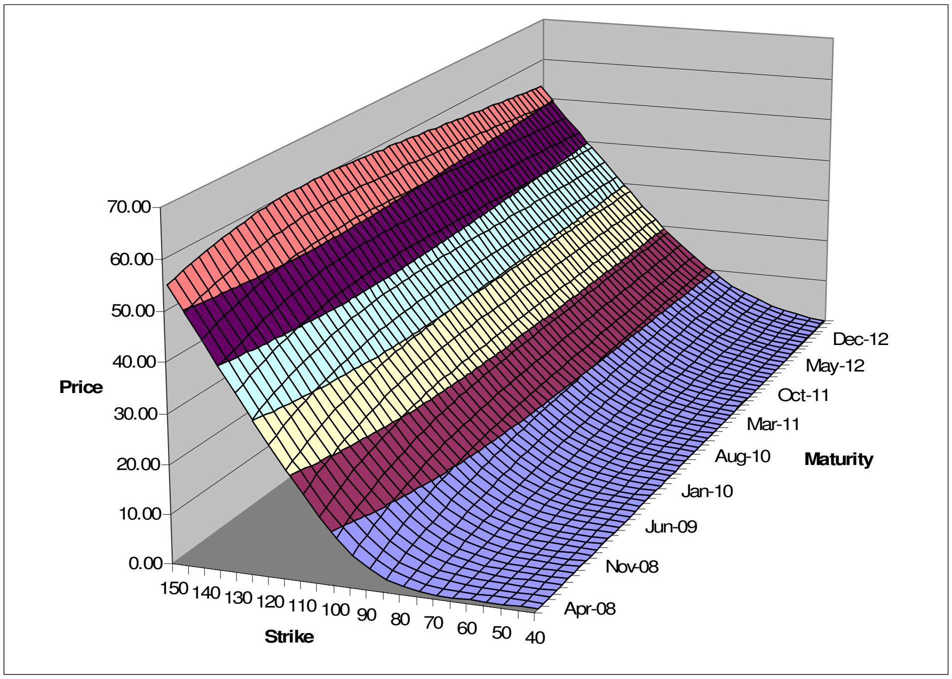


Figure 4: Example market Asian call option (a) and put option (b) price surfaces

The transition probabilities can be calculated once the state prices have been determined. We specify a no-arbitrage relationship such that we have:

Period pure discount bond price consistency:

$$e^{-r_i \Delta t}\left(p_{d,i,j} + p_{m,i,j} + p_{u,i,j}\right) = e^{-r_i \Delta t}$$

(6)

Spot price consistency:

$$F(t_i, S_{i,j}, t_{i+1}) = \left(p_{d,i,j} S_{i+1,j-1} + p_{m,i,j} S_{i+1,j} + p_{u,i,j} S_{i+1,j+1}\right)$$

(7)

Discrete forward equation for state prices:

$$Q_{i+1,j+1} = e^{-r_i \Delta t}\left(p_{d,i,j+2,} Q_{i,j+2} + p_{m,i,j+1} Q_{i,j+1} + p_{u,i,j} Q_{i,j}\right)$$

(8)

Where $F(t_i, S_{i,j}, t_{i+1})$ is the forward price at node (i, j) for maturity t_{i+1}.

As with the state prices, the above transition probabilities can be determined by solving the set of linear equations (6), (7) and (8) for each node *j* at a particular time step i via standard linear algebra techniques.

Once the transition probabilities have been determined, the local volatility at each node $\sigma_{i,j}$ can be obtained by assuming that the spot price returns over the tree time steps are approximately normally distributed which gives:,

$$E\left[\Delta x_{i,j}\right] = p_{u,i,j}\Delta x_{u,i,j} + p_{m,i,j}\Delta x_{m,i,j} + p_{d,i,j}\Delta x_{d,i,j}$$

$$\sigma^2{}_{i,j}\Delta t = p_{u,i,j}(\Delta x_{u,i,j})^2 + p_{m,i,j}(\Delta x_{m,i,j})^2 + p_{d,i,j}(\Delta x_{d,i,j})^2 - E\left[\Delta x\right]^2 \quad (10)$$

The local volatility surface captures the behaviour of the market option prices. Figure 5 provides an example of a local volatility surface obtained through calibration of a trinomial tree to Asian call and put option prices. Notice that for ATM options the volatilities are the same as for the spot volatility curve shown in Figure 1. Further, the local volatility is shown to increase for strike levels that are increasingly “in the money” or “out of the money”.

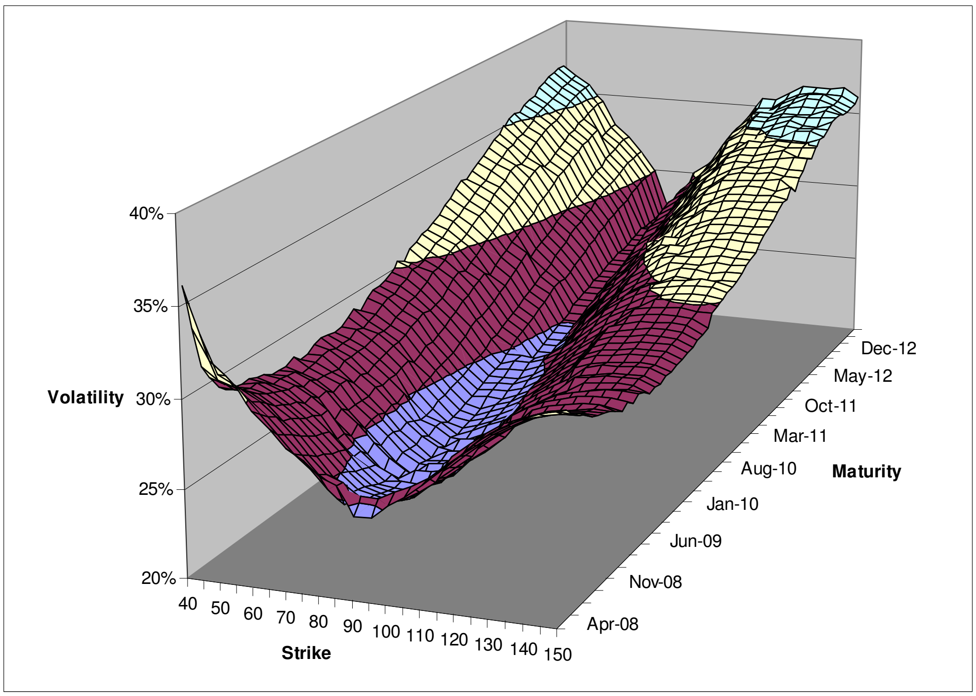


Figure 5: Local volatility surface obtained by calibrating a trinomial tree to Asian options.

References

Barone-Adesi, G and R. E. Whaley, 1987, “Efficient Analytical Approximation of American Option Values”, Journal of Finance XLII, No. 2.

Breslin, J., Clewlow, L., Strickland, C., and van der Zee, D, 2008, “Swing Contracts: Take it or Leave it”, *EnergyRisk* , February.

Clewlow, L. and C. Strickland, 1998, *Implementing Derivatives Models*, John Wiley, London.

Clewlow, L. and C. Strickland, 1999, “Valuing Energy Options in a One-Factor Model Fitted to Forward Prices”, Working Paper, University of Technology, Sydney.

Clewlow, L, and C. Strickland, 2000, “*Energy Derivatives: Pricing and Risk Management*”, Lacima Publications.

Derman, E., I. Kani and N. Chriss, 1996, “*Implied Trinomial Trees of the Volatility Smile*”, Quantitative Strategies Research Notes, Goldman Sachs.

Levy, E., "Pricing European Average Rate Currency Options", Journal of International Money and Finance, 14, 474-491. (1992)

Schwartz, E.S., 1997, "The Stochastic Behaviour of Commodity Prices: Implications for Valuation and Hedging", *The Journal of Finance*, Vol LII, No. 3, July, pp 923-973.

Turnbull, S. M. Wakeman, L. M., "A Quick Algorithm for Pricing European Average Options", Journal of Financial and Quantitative

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