Innovations in risk analytics

Implied trees: valuing exotic options

Authors:

Jim Clark
Les Clewlow
Chris Strickland

Lacima Group

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In the first article of this series[1], we outlined an approach for calibrating a local volatility surface single factor model to market prices of average price (Asian) options. In this article we will show how to value exotic options, using the calibrated local volatility surface, consistent with the market prices.

The reader will recall from the previous article that the basic idea of implied tree approaches is to construct a trinomial tree that is consistent with, whether exactly or to some approximation, currently traded energy derivatives prices. Once the local volatility function is determined, the trinomial tree can then be used to price any other derivative on the same underlying energy asset with the same or earlier maturity. In particular, market implied information embodied in the constructed tree enables the pricing of OTC and exotic options consistent with the prices of all liquid options with the same underlying.

In this article we will calibrate the tree to the market prices of fixed strike Asian options and then show how to price floating strike Asian options as our example of an alternative derivative.

As with the fixed strike Asian option, the floating strike Asian option is path dependent – the value of the option at any node in the implied tree is dependent on the path the asset price took to reach that node. Table 1 summarises the different types of Asian options outlined in this article, together with the mathematical definition of their pay-off. In the case of a fixed strike option, the final asset price is the arithmetic average of the spot prices observed on the set of fixing dates t_i where $i = 1,...,m$, and the strike price is constant. For a floating strike option the final *strike* price is the arithmetic average of the spot prices, and the final asset price is the final spot price.

Fixed strike Asian call	$\max\left(0, \frac{S_{t1} + S_{t_2} + ... + S_{t_m}}{m} - K\right)$
Fixed strike Asian put	$\max\left(0, K - \frac{S_{t_1} + S_{t_2} + ... + S_{t_m}}{m}\right)$
Floating strike Asian call	$\max\left(0, S_T - \frac{S_{t1} + S_{t_2} + ... + S_{t_m}}{m}\right)$
Floating strike Asian put	$\max\left(0, \frac{S_{t_1} + S_{t_2} + ... + S_{t_m}}{m} - S_T\right)$

Table 1: **Different Asian Option Types**

[1] Clark et al, Energy Risk August 2008

In our last article we showed how to construct a trinomial tree consistent with an initial forward curve, the market prices of fixed strike Asian options, and with the assumption of a mean reverting model for the spot price. In order to help with the pricing discussion that follows, we repeat some of the notation here. Each node in the resulting tree is identified by a pair of integers (i, j) where $i = 0,\ldots,N$ is the time step and $j = -i,\ldots,i$ is the level of the asset price relative to the initial asset price. At node (i, j) the date is t_i where we assume $t_0 = 0$ and the spot price is $S_{i,j}$ with $S_{0,0} = S(0)$, the initial spot price. Each node has three branches which connect with nodes at the next time step and each branch has an associated transition probability which we denote $p_{u,i,j}$, $p_{m,i,j}$, and $p_{d,i,j}$ for the 'up', 'middle', and 'down' probabilities respectively. The resulting trinomial tree is represented graphically in Figure 1.

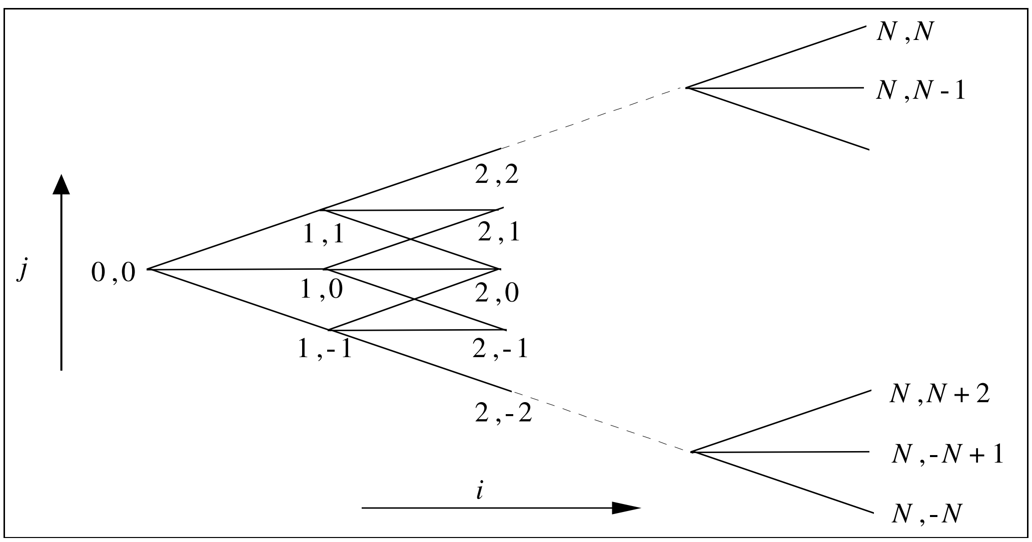

Figure 1: **Nodes in the trinomial tree**

Note that in practice we may truncate the tree to reflect the very small probabilities at the edges of the tree that arise due to the mean reverting nature of the process we are using. In the construction of the trinomial tree we utilized the concept of state prices, or Arrow-Debreu securities, which form the building blocks of all securities. These state prices were denoted by $Q_{i,j}$ which define, at time 0, the price of a security that pays $1 if node (i, j) is reached, and $0 otherwise.

For the pricing of exotic options from the tree that have a European (single exercise at maturity) payoff, we can utilize either the state prices or the transition probabilities generated for each node of the tree. For example the price today

($t=0$), $C(0)$ of any European claim with a payoff $C(S)$ as a function of the spot price S at the maturity time step N in the tree is given by:

$$C(0)=\sum_j Q_{N,j}C(S_{N,j})$$

(1)

where the summation takes place across all of the nodes j at time N.

That is, for a European call option on the spot price, with strike price *K*,

$$C_{EuroCall}\left(0;K,T_N,S_{N,j}\right)=\sum_j Q_{N,j}\max\left(0,S_{N,j}-K\right)$$

(2)

For the same European claim $C(S)$, we can step back through the tree computing the discounted expectation via the relationship,

$$C_{i,j}=C(S_{i,j})=e^{-r_i\Delta t}\left(p_{u,i,j}C_{i+1,j+1}+p_{m,i,j}C_{i+1,j}+p_{d,i,j}C_{i+1,j-1}\right)$$

(3)

where r_i denotes the 1 period interest rate and Δt is the time step size per period. The option payoff for each node at maturity $C_{N,j}$ is first calculated, and the above discounted expectation is applied to give the price today of the option $C_{0,0}$. The equivalence between equations (1) and (3) can be seen from equation (8) of our last Masterclass series from which we can interpret the Arrow-Debreu securities as discounted probabilities.

If the payoff is based on the price of a futures contract then we have,

$$C_{EuroCall}\left(0;K,T_N,F(T_N,S_{N,j},T_M)\right)=\sum_j Q_{N,j}\max\left(0,F(T_N,S_{N,j},T_M)-K\right)$$

(4)

Where $F(T_N,S_{N,j},T_M)$ defines the forward price at node (N,j) for maturity at time $N+M$, This forward price can be calculated from the spot price, $S_{N,j}$, and the initial forward curve.

In order to illustrate some properties for the tree we priced a standard European at-the-money 3 month call option on the spot price with increasing numbers of steps per year (from monthly time steps down to daily time steps). For this option we can derive an analytic value, allowing us to comment on the convergence properties of the numerical technique. Figure 2 plots this convergence.

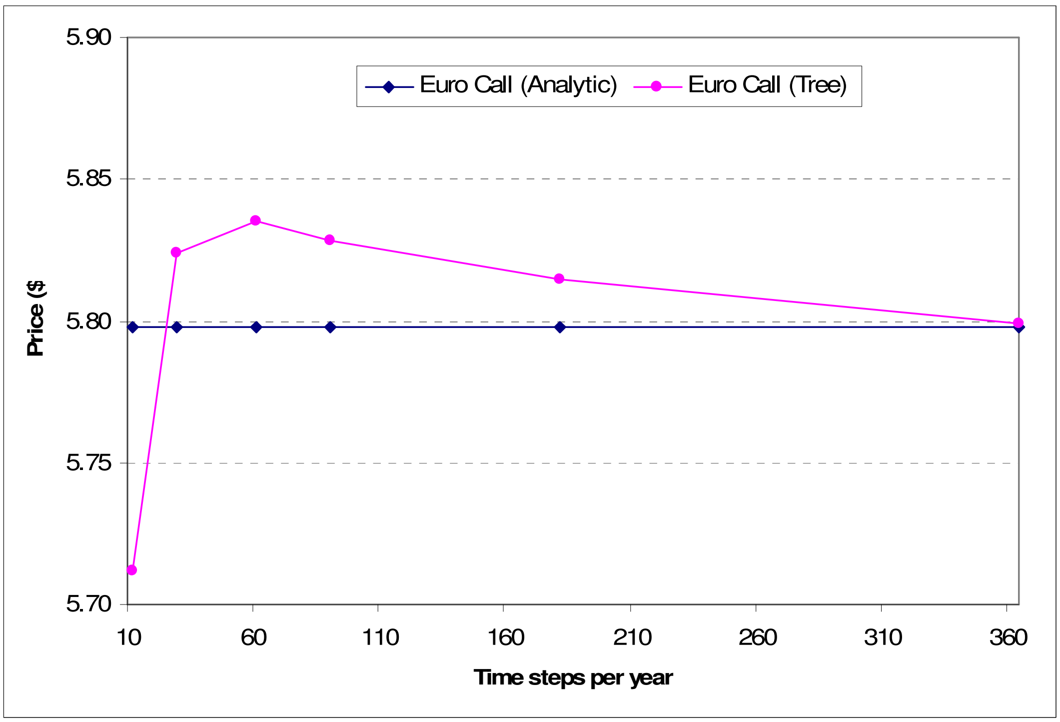

Figure 2: **Price convergence of standard call option as number of steps in tree increases**

Figure 2 shows the tree prices the standard option to decimal place accuracy for daily time steps, and so we use daily time steps for the remainder of the analysis.

Fixed and Floating strike Asian options

There are a number of steps to follow in order to price an Asian option whose payoff depends on some function $G(.)$ of either the path of the spot price or of the forward price curve. Firstly, we need to determine the range (i.e. the minimum and maximum) of possible values of $G(.)$ which can occur for every node in the tree. Secondly, we must choose an appropriate set of $n_{i,j}$ values of $G(.)$ between the minimum and maximum values for each node. In choosing $n_{i,j}$ one must consider the distributional properties of the function $G(.)$. Increasing the number of range values increases the accuracy of the pricing method. However, greater accuracy comes at the expense of increased computational effort in determining a solution using the tree based implementation. The final steps in the procedure require setting the value of the option at maturity at every node and for every value of $G(.)$, and then stepping back through the tree computing discounted expectations at every node and for every value of $G(.)$ (analogous to the steps described in the previous section).

The payoff at maturity for an Asian call option with a fixed strike price is,

$$C_{N,j,k} = \max\left(0, G_{N,j,k} - K\right) \tag{5}$$

where $G_{N,j,k}$ is the average of the k^{th} spot price path (or forward curve path) at node level j, and $k = 1,...,n_{i,j}$. The payoff at maturity for an Asian call option (on spot) with a floating strike price is,

$$C_{N,j,k} = \max\left(0, S_{N,j} - G_{N,j,k}\right) \tag{6}$$

where the corresponding payoff for the option on forward contract follows. For a detailed explanation of this procedure refer to Clewlow et al [1998] or Clewlow et al [2000].

One of the key determinants of the accuracy of the Asian option implementation in the tree framework is the number of averages ($n_{i,j}$) defined at each node. In order to investigate this, we price a European exercise, fixed strike Asian call with 3 months maturity and where we utilize daily averaging over the last month to determine the option payoff. For this option we can compare the tree based value with a closed form approximation, and our analysis is presented in figure 3.

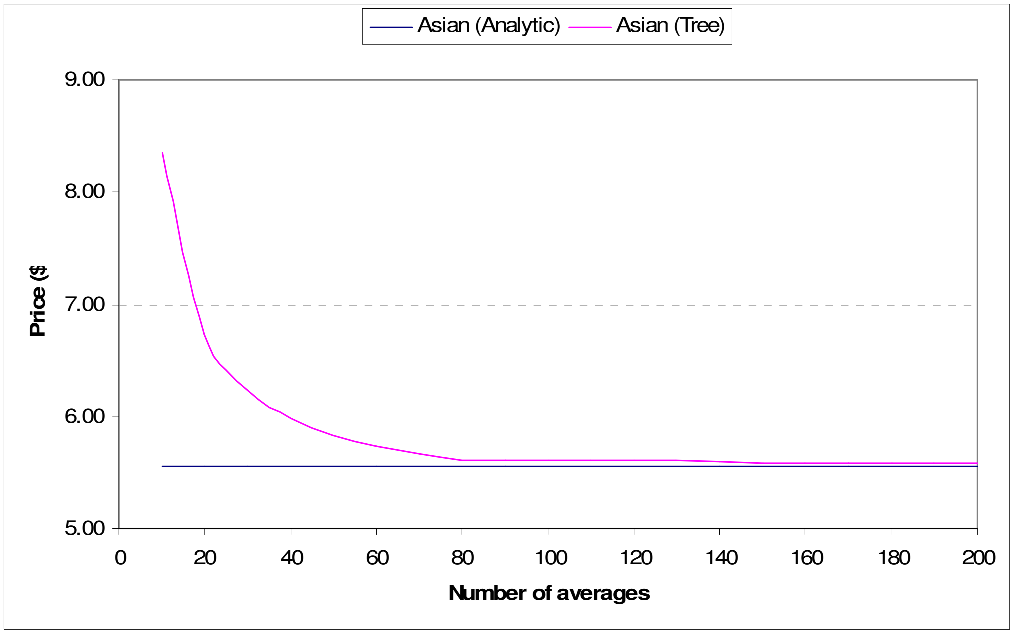

Figure 3: **Price convergence of Asian fixed strike call option for an increasing number of averages**

We can see from figure 3 that the value of the Asian option from the tree converges to the analytical approximation as we increase the number of averages. In our subsequent analysis we fix the number of averages to be 200, however, in practical implementations of Asian option pricing in trees we note that the nodes which lie on the upper and lower edges of the tree have only one path which reaches them and therefore there will only be one value of the average. We can therefore adjust the number of averages we hold in the tree dependent on the position of the node in the tree. The largest range of values will occur in the central section of tree. The number of averages increases exponentially with the number of time steps since there is a different average for each path. However it is impractical to work with an exponentially increasing number of averages because the computational cost would be too high. Therefore we approximate with a number of values which increases linearly with the number of time steps but also decreases linearly from the central nodes of the tree down to one at the edges of the tree. We can choose $n_{i,j}$ to be given by

$$n_{i,j} = 1 + \beta\left(i - abs(j)\right) \qquad (7)$$

so that $n_{i,j}$ will always be one at the edges of the tree (*j*=*i* and *j*=-*i*) and $1+\beta i$ in the centre of the tree. In this way we can increase □ to increase the accuracy of the approximation by considering more values of the average.

The convergence analysis just performed was under the assumption of no skew in the volatility surface. In the remainder of our analysis we generate a local volatility skew by calibrating the tree to a market oil forward curve from the end of 2007, as well as to a set of fixed strike Asian call and put options. The resulting local volatility surface is presented in figure 4.

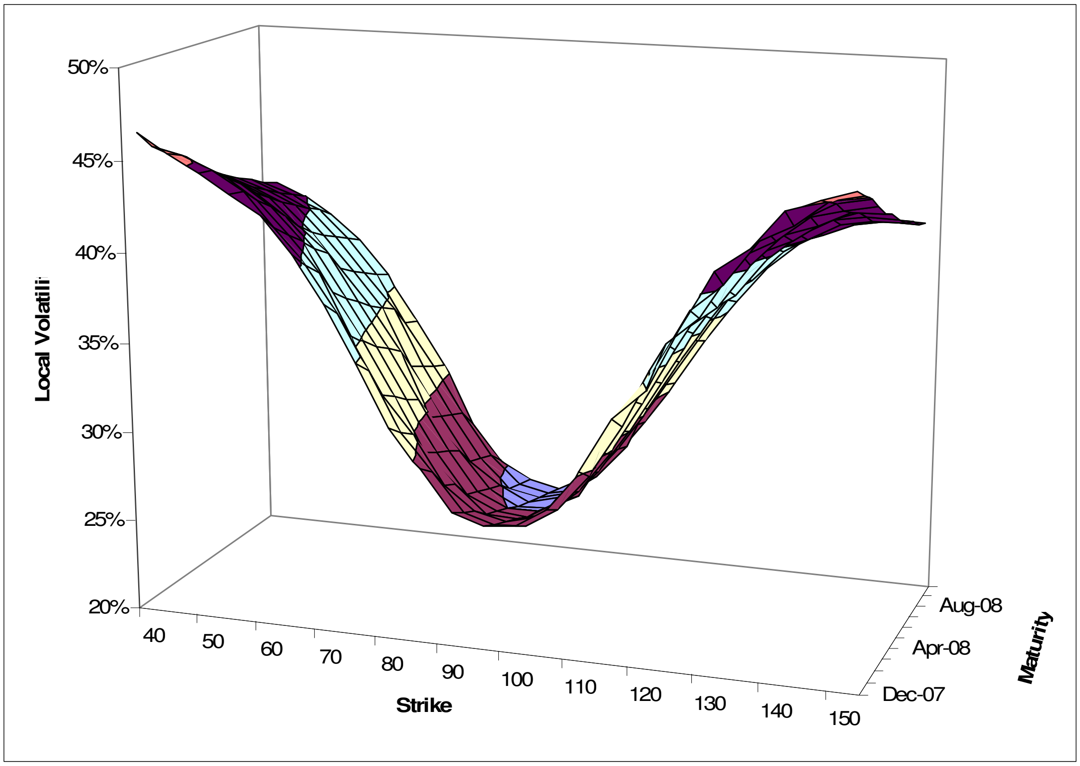

Figure 4: **Local volatility surface with smile**

Figures 5a & 5b present the values and price differences for the 3 month Asian option described earlier with the option prices with & without skew. The prices for both Asian call options with (Tree skew) and without (Tree SF) skew are presented for a variety of strikes in figure 5a. The differences in price between the two Asian call options are shown in figure 5b. Observe that Asian call options with a positive smile are priced higher than for those Asian call options without skew. This is to be expected due to a greater average volatility for the skewed surface. This price difference is greater for at-the-money (ATM) and near-the-money options ($0.10 to $0.17), and decreases to zero for deep in-the-money (ITM) and out-of-the-money (OTM) options.

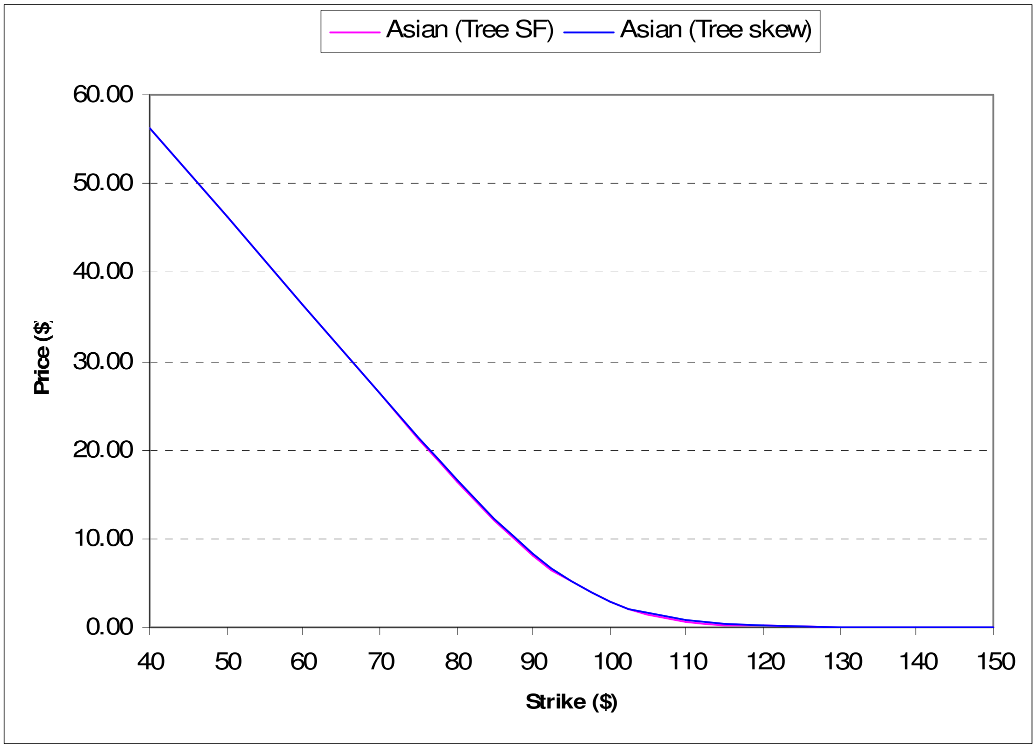

Figure 5a: **Comparison of fixed strike Asian call option prices (with and without skew)**

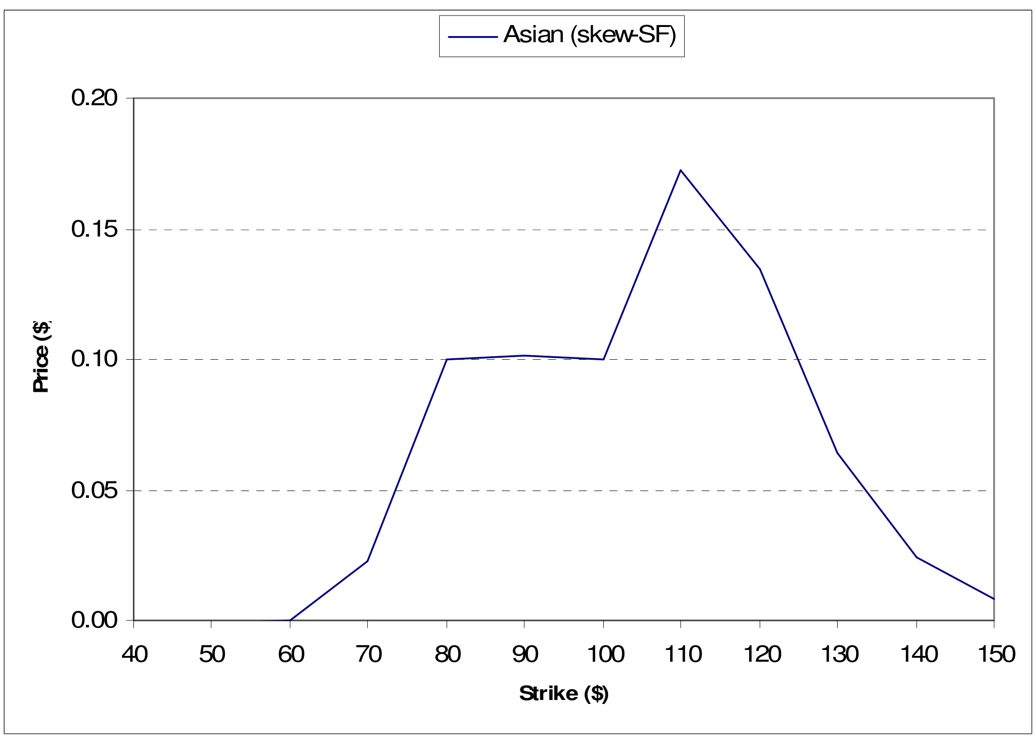

Figure 5b: **Price difference for fixed strike Asian call option prices (with and without skew)**

Finally we compare the prices of fixed and floating strike Asian options. The floating strike Asians are used as an example another derivative once the tree has been calibrated to the fixed strike Asians. Figure 6 shows a plot of fixed strike ATM and floating strike Asian call option prices for a range of maturities. Both options utilize daily averaging over the last month. The fixed strike Asian ATM option increases in value for longer maturities, reflecting the increasing time value of the option struck at a fixed strike price today. In contrast, the floating strike option value does not change much over the same maturity horizon given that the strike price is a moving average of the spot prices prior to the maturity date.

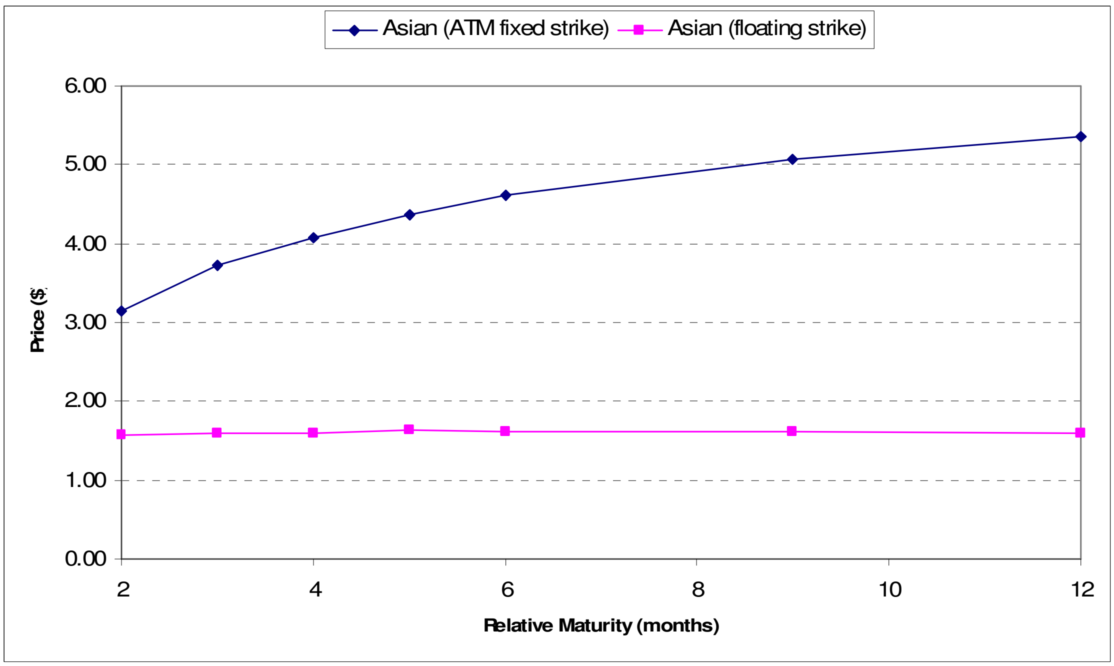

Figure 6: **Fixed strike (ATM) and floating strike Asian call option prices for a range of maturities**

The implied tree we have constructed is a discrete time and state approximation to the continuous time risk neutral process represented by the following SDE

$$\frac{dS(t)}{S(t)} = \alpha[\theta(t) - \ln S(t)]dt + \sigma(S,t)dz(t)$$

(8)

In this article we have described how an implied tree could be used to price exotic options. We thus have a way of pricing exotic options consistent with the market prices of standard European options and therefore consistent with the implied volatility smile. Furthermore we can use the implied tree to compute hedge sensitivities in order to dynamically hedge the exotics options with the standard options and underlying asset. However note that if equation (7) is not a good representation of the behaviour of the underlying energy price, for example if the asset price behaviour involves jumps or stochastic volatility, then hedging strategies based on the implied tree may not perform very well. Furthermore dynamic hedging with standard options is problematic because of the more complex structure of the instruments and their costs. A partial solution to both of these problems is the concept of static replication. The idea is to find a portfolio of standard options which can be acquired today which will replicate the exotic option in all possible future states of the world. The details of this approach will be the subject of its own article later in our Masterclass series.

References

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